Chapter 4

Applications of the Derivatives

Objectives: By the end of this Chapter students should be able to:

- **Use the derivative to locate the minimum and maximum values of a function on an interval**
- **Understand and use Rolle's Theorem and The Mean Value Theorem**
- **Use the Derivative tests to locate local minimum and local maximum**
- **Use the derivative tests to find intervals where a function is increasing and decreasing**
- **Find horizontal Asymptotes**
- **Sketch graphs using ideas of calculus**
- **Solve optimization problems**
- **Use approximation techniques**

Increasing and Decreasing Functions

Objectives: By the end of this section students should be able to:

- Find intervals where a function is increasing or decreasing from graph
- Find critical numbers and critical points
- Find intervals where a function is increasing or decreasing using the derivative

Definition (Increasing and Decreasing functions)

a) A function f is said to be an **increasing** function on an interval **I**, if for all x_1 **and** x_2 in **I**,

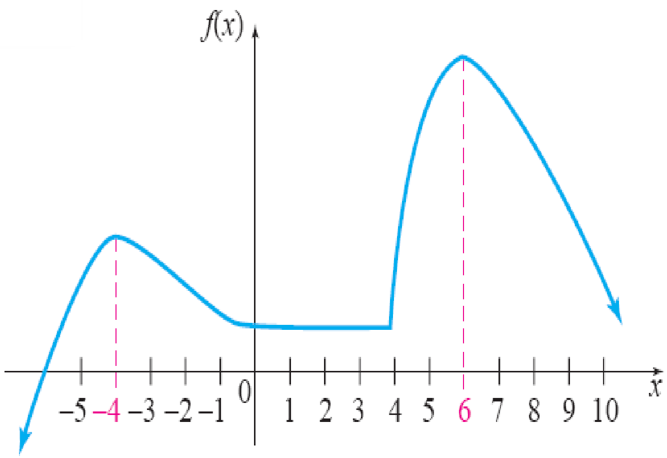
$x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

b) A function f is said to be an **decreasing** function on an interval **I**, if for all x_1 **and** x_2 in **I**,

$x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

c) If the value of a function f **does not** change in an interval **I**, then f is a **constant** on **I**

Example 1: For the graph shown below: find intervals were the function is increasing, decreasing or a constant



Example 2: Show that the function a) $f(x) = 5x + 4$ is an increasing function

b) $f(x) = -3x + 2$ is a decreasing function

c) $f(x) = x^2 + 3x - 5$ is neither increasing or decreasing

Worked Example: Show, using the definition, $f(x) = x^3$ is an **increasing function**

Solution: We want to show $x_1 < x_2$ implies that $f(x_1) < f(x_2)$ for all real numbers x_1 *and* x_2.

So, start with $x_1 < x_2$ which implies $x_2 - x_1 > 0$ which also gives $(x_2 - x_1)^3 > 0$.

But $(x_2 - x_1)^3 > 0$ and $(x_2 - x_1)^3 = x_2{}^3 - 3x_2{}^2x_1 + 3x_1{}^2x_2 - x_1{}^3$

implies that $x_2{}^3 - 3x_2{}^2x_1 + 3x_1{}^2x_2 - x_1{}^3 > 0$ Moving some terms to the right we get:

$x_2{}^3 - x_1{}^3 > 3x_2{}^2x_1 - 3x_1{}^2x_2$ factoring gives $x_2{}^3 - x_1{}^3 > 3x_2x_1(x_2 - x_1)$.

Now we consider **different cases**:

1) **Both** x_1 *and* x_2 **negative** numbers

   If $x_1 < 0$ *and* $x_2 < 0$ *then* $x_1x_2 > 0$, since $x_2 - x_1 > 0$ we get that $3x_2x_1(x_2 - x_1) > 0$

   This gives: $x_2{}^3 - x_1{}^3 > 0$**; proving the statement**

2) **Both** x_1 *and* x_2 **positive** numbers

   If $x_1 < 0$ *and* $x_2 < 0$ *then* $x_1x_2 > 0$, since $x_2 - x_1 > 0$ we get that $3x_2x_1(x_2 - x_1) > 0$

   This gives: $x_2{}^3 - x_1{}^3 > 0$**; proving the statement**

3) x_1 **is zero or negative** and x_2 **positive**

   $x_1 \leq 0$, implies that $x_1{}^3 \leq 0$ and $x_2 > 0$ implies that $x_2{}^3 > 0$, thus obviously $x_2{}^3 > x_1{}^3$

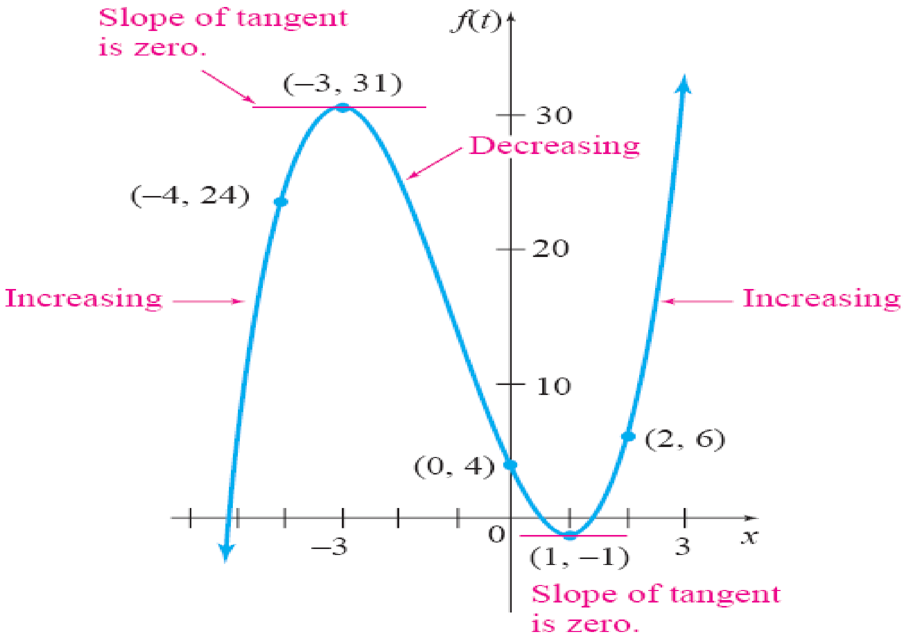
From **Cases 1), 2), and 3)** the statement $x_1 < x_2$ implies $f(x_1) < f(x_2)$ follows, thus the function $f(x) = x^3$ is increasing.

Definition (Test for Increasing and Decreasing Functions)

Suppose f is a function that is continuous on the interval **[a, b]** and has a derivative at each point in an open interval **(a, b).**

- If $f'(x) > 0$ for each x in the interval **(a, b),** then f is **increasing** on **[a, b].**
- If $f'(x) < 0$ for each x in the interval **(a, b),** then f is **decreasing** on **[a, b].**
- If $f'(x) = 0$ for each x in the interval **(a, b),** then f is **constant** on **[a, b]**.

Example 1: For the graphs shown below: find intervals were the function is increasing, decreasing or a constant



Definition (Critical Number)

The **critical numbers** for a function ***f*** are those numbers **c** in the domain of ***f*** for which $\boldsymbol{f'(c)=0}$ or $\boldsymbol{f'(c)}$ **does not exist**. **A critical point** is a point whose **x-coordinate** is the critical number **c**, and whose **y-coordinate** is ***f*** **(c), so (*c*, *f*(*c*))** is a critical **point.**

Example 2: Find the **critical number**(s).

a) $y = x^3 + 3x^2 - 9x + 4$

b) $y = \dfrac{x+2}{x+1}$

c) $y = -\sqrt{x+2}$

Example 3: Find the **critical points** for each of the following.

a) $f(x) = x^3 + 3x^2 - 9x + 4$

b) $f(x) = x + \ln|x|$

Sign Chart for finding intervals where a function is increasing and decreasing

Steps to follow

1. Find the derivative
2. Find the critical numbers.
3. Place critical numbers and undefined values on number line.
4. Test values in each interval. (**Plug into** $\boldsymbol{f'(x)}$ **)**
5. Increasing intervals $\boldsymbol{f'(x) > 0}$
   Decreasing intervals $\boldsymbol{f'(x) < 0}$

Example 4: Find the intervals where the function is increasing and decreasing.

a) $y = x^2 - 3x + 1$ b) $y = 3x + 2$

c) $f(x) = -\sqrt{x+2}$ d) $f(x) = \dfrac{x+2}{x+1}$

Example 5: Find the interval(s) where the function is increasing.

a) $f(x) = x^4 + 4x^3 + 4x^2 + 1$

b) $f(x) = x^3 + 3x^2 - 9x + 4$

4.1 Extrema on an Interval and First Derivative Test

Objectives: By the end of this section students should be able to:

- **Find extrema of a function on an interval**
- **Find local or relative extrema of a function**

Definition of Extrema

Let *f* be a function defined on an interval I containing the number c

1) $\boldsymbol{f(c)}$ is the minimum of $\boldsymbol{f}$ on I if $\boldsymbol{f(c) \leq f(x)}$ for all ***x in I***
2) $\boldsymbol{f(c)}$ is the maximum of $\boldsymbol{f}$ on I if $\boldsymbol{f(c) \geq f(x)}$ for all ***x in I***

The minimum or maximum values of a function f on an interval I are the extreme values, or extrema (singular extremum) of the function on the interval

The Minimum or maximum values of a function on an interval are also called the absolute minimum or the absolute maximum of the function on the interval.

Theorem 4.1 (The Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval

Guide lines for finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval [a, b], use the following steps.

1) Find the critical numbers of f in (a, b)
2) Evaluate f at each critical numbers in (a, b)
3) Evaluate f at each end points of [a, b]
4) The list of these values is the minimum; the greatest is the maximum

Example 1: Find the minimum and maximum of

a) $f(x) = x^2 - 4x + 2$ on the closed interval $[-1, 3]$
b) $f(x) = x^3 - 2x^2 - 3x - 2$ on the closed interval $[-1, 3]$
c) $f(x) = 3x^4 - 4x^3$

Practice Problems:
Page 209, Exercises 4.1: 1 – 5, 11, 13, 15, 17, 23, 24, 27, 29, 33, 37, 41, 43

4.2 Rolle's Theorem and The Mean Value Theorem

Theorem 4.3: Rolle's Theorem

Let f be continuous on the closed interval **[a, b]** and differentiable on the open interval **(a, b).** If $f(a) = f(b)$, then there is at least one number **c** in **(a, b)** such that $f'(c) = 0$.

Proof: Refer to the Book

Example 2: Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

Theorem 4.4: The Mean value Theorem

If f is continuous on the closed interval **[a, b]** and differentiable on the open interval **(a, b),** then there exists a number **c** in **(a, b)** such that: $f(x) = \frac{f(b) - f(a)}{b - a}$.

Proof: The equation of the secant line that passes through the point (a, f(a)) and (b, f(b)) is

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Let $g(x) = f(x) - y$

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

Then, it follows that g is differentiable on (a, b) with $g(a) = g(b) = 0$, then by Rolle's Theorem there is a number **c** in **(a, b)** such that $g'(c) = 0$; From which The Mean Value Theorem Follows

Practice Problems:
Page 216, Exercises 4.2: 1 – 4, 11 – 14, 22 – 26, 43 – 46, 49, 51, 54, 55

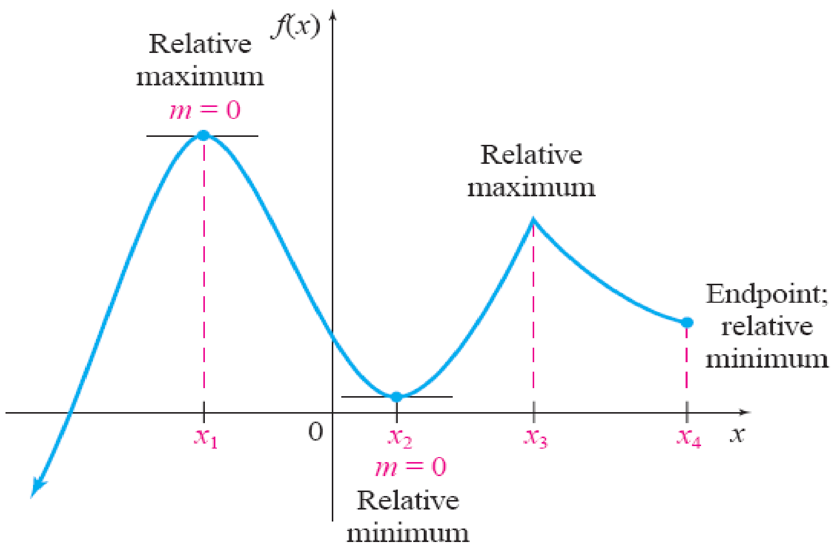
Relative or Local Extrema

Relative Maximum or Minimum

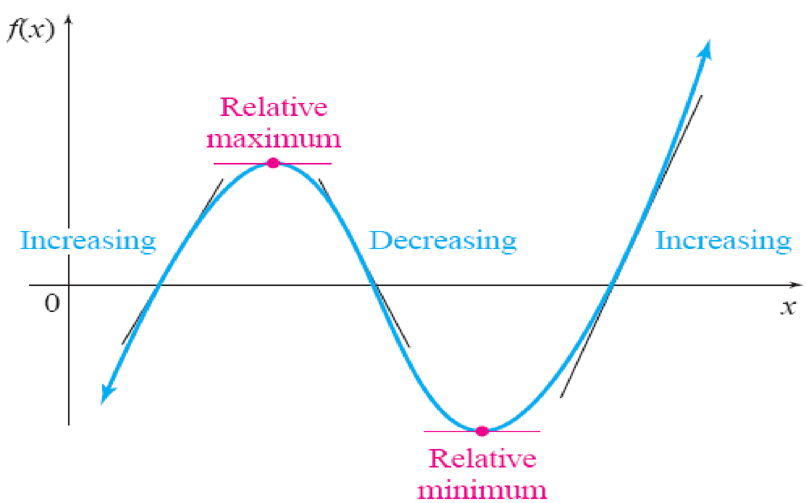
Let *c* be a number in the domain of a function *f*.

1. Then the number $\boldsymbol{f(c)}$ is a **relative** (or local) **maximum** for ***f*** if there **exits** an **open** interval **(a, b)** containing ***c*** such that $\boldsymbol{f(x) \leq f(c)}$ for all ***x*** in **(a, b).** We also say f has a relative maximum at **(c, f(c))**
2. The number $\boldsymbol{f(c)}$ is a **relative** (or local) **minimum** for ***f*** if there exits an **open** interval **(a, b)** containing ***c*** such that $\boldsymbol{f(x) \geq f(c)}$ for all ***x*** in **(a, b).** We also say ***f*** has a relative minimum at **(c, f(c))**

Let's look at some graphs to get an idea at what we are looking for.

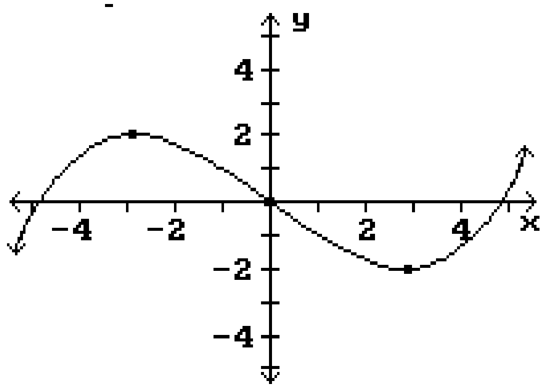


Note: Here x_1, x_2, *and* x_3 are critical numbers and at this point we have extreme values.



Note: Where we have the relative extreme values, notice how the function changes from increasing to decreasing or from decreasing to increasing. That is, notice how the **derivative changes signs**.

Example 1: a) Find the relative extrema.



b) Where does the extremum value occur?

Theorem 4.2

If f has a relative minimum or a relative maximum at $x = c$, then c is a **critical number** of f.

Proof: Refer to the text

Assume that f has a relative extremum at the point $x = c$.

We want to show the point $x = c$ is a critical point. At the point $x = c$ either the derivative **exists** or **does not exist**. Two cases to consider

Case 1) If the derivative of f at $x = c$ **does not exist**, then by definition $x = c$ is a **critical number**.

Done

Case 2) If the derivative of f at $x = c$ **does exist,** then either $f'(c) > 0$, or $f'(c) < 0$, or $f'(c) = 0$.

First assume $f'(c) > 0$, then there is an open interval **(a, b)** containing c such that

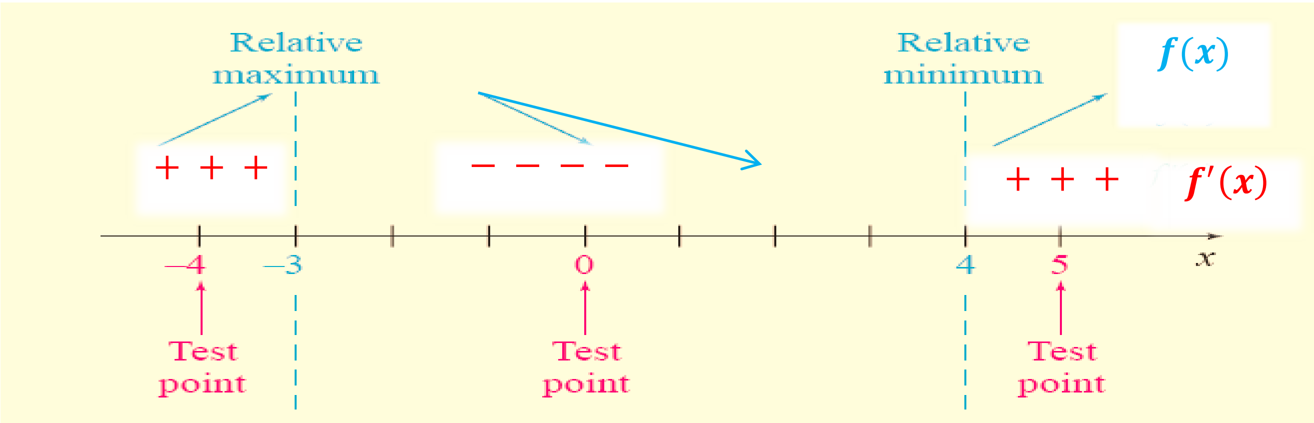
$\frac{f(x)-f(c)}{x-c} > 0$ for all x in **(a, b)** which implies that the numerator $f(x) - f(c)$ and the denominator $x - c$ are of the **same sign**

So, in the case $x < c$, we have $f(x) < f(c)$; **and** in the case $x > c$, we have $f(x) > f(c)$; which contradicts the fact that $f(c)$ is a **local extreme.** Thus, $f'(c) > 0$ is not possible; a similar argument will show that $f'(c) < 0$ is **not** possible either. This left us with the only possibility, which is $f'(c) = 0$, proving the **Theorem**

Example 2: $f(x) = x^3 - 3x^2$

First Derivative Test

We can use the first derivative to determine where the relative maximum or relative minimum occurs. Look at the figure below:



Note: Where the Derivative changes sign the local extremum happens

Theorem 4.6 (The First Derivative Test)

Let **c** be the **critical number** of a function f that is continuous on an open interval **I** containing **c**. If f is differentiable on the interval, except possibly at **c,** then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from **negative** to **positive** at **c**, then f has relative **minimum** at $(c, f(c))$.
2. If $f'(x)$ changes from **positive** to **negative** at **c**, then f has relative **maximum** at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of **c** or negative on both sides of **c**, then $f(c)$ is neither a relative minimum nor a relative maximum.

Proof: 2. $f'(x)$ changes sign **positive** to **negative** at **c,** implies there are numbers **a** and **b** in **I** such that

$f'(x) > 0$ for all x in **(a, c);** that is, f is increasing on **(a, c)**.

Which implies $f(x) \leq f(c)$ for all x in **(a, c)** . . . . . **(*)**

$f'(x) < 0$ for all x in **(c, b);** that is, f is decreasing on **(c, b)**.

Which implies $f(c) \geq f(x)$ for all x in **(c, b)** . . . . . **(**)**

From **(*) and (**)** it follows that $f(x) \leq f(c)$ for all x in **(a, b);** that means, f has a **relative maximum** at $(c, f(c))$.

1. Is proved in a similar manner

Example 3: **Using the First Derivative Test**, find the value of any relative extrema.

a) $y = x^2 - 3x + 1$

b) $f(x) = x^4 + 4x^3 + 4x^2 + 1$

c) $f(x) = \dfrac{1}{x^2 - 4}$

d) $f(x) = -x^2 + 12x - 8$

e) $f(x) = x^2 e^x - 3$

Example 4: (Page 223 of Text) Find the relative extrema of $\boldsymbol{f(x) = (x^2 - 4)^{2/3}}$

Solution: There are three critical numbers for this function $\boldsymbol{x = -2}$, $\boldsymbol{x = 2}$, and $\boldsymbol{x = 0}$

Interval	$\boldsymbol{-\infty < x < -2}$	$\boldsymbol{-2 < x < 0}$	$\boldsymbol{0 < x < 2}$	$\boldsymbol{2 < x < \infty}$
Test Value	$\boldsymbol{x = -3}$	$\boldsymbol{x = -1}$	$\boldsymbol{x = 1}$	$\boldsymbol{x = 3}$
Sign of $\boldsymbol{f'(x)}$	$\boldsymbol{f'(-3) < 0}$	$\boldsymbol{f'(-1) > 0}$	$\boldsymbol{f'(1) < 0}$	$\boldsymbol{f'(3) > 0}$
Conclusion	**Decreasing**	**Increasing**	**Decreasing**	**Increasing**

Example 5: (Page 224 of Text, Reading) Find the relative extrema $\boldsymbol{f(x) = \frac{x^4+1}{x^2}}$

Example 6: A differentiable function $\boldsymbol{f}$ has one critical number at $\boldsymbol{x = 5}$. Identify the relative extrema of $\boldsymbol{f}$ at the critical number if $\boldsymbol{f'(4) = -2.5}$ and $\boldsymbol{f'(6) = 3}$

Example 7: Consider $\boldsymbol{f(x) = axe^{bx^2}}$ **.** Find $\boldsymbol{a}$ and $\boldsymbol{b}$ such that the relative maximum of $\boldsymbol{f}$ is $\boldsymbol{f(4) = 2}$

Practice Problems:
Page 226, Exercises 4.3: 1 – 8, 9, 11, 15, 17, 19, 23, 25, 27, 33, 35, 43, 45, 49, 55, 61, 63, 65

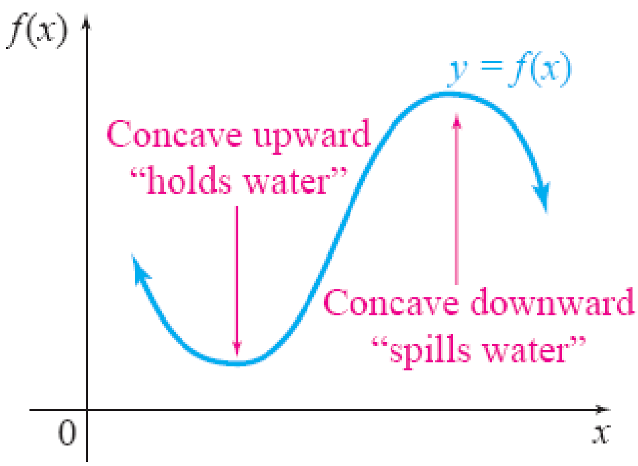
4.4 Concavity and the Second Derivative Test

Objectives: By the end of this section students should be able to:

- Determine intervals on which a function is concave upward or concave downward
- Find inflection points
- Apply the Second Derivative Test to find relative extrema of a function

Definition (Concavity)

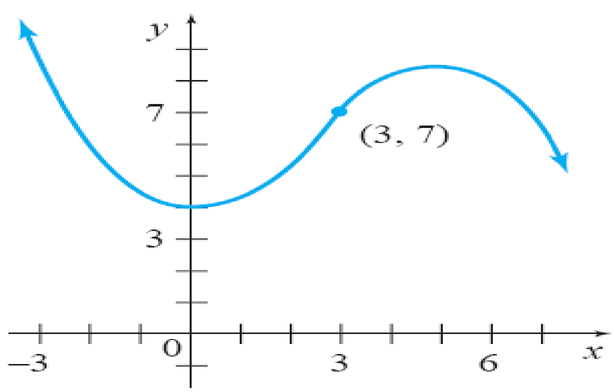
Let $\boldsymbol{f}$ be differentiable on an open interval **I**. The graph of f is **concave upward** on **I** if $\boldsymbol{f'}$ is **increasing** on **I** and **concave downward** if $\boldsymbol{f'}$ is **decreasing** on the interval. **See figure**



Definition (Point of Inflection)

A point $(\boldsymbol{c}, \boldsymbol{f}(\boldsymbol{c}))$ is called an inflection of the graph of a function $\boldsymbol{f}$ at $(\boldsymbol{c}, \boldsymbol{f}(\boldsymbol{c}))$ the concavity of $\boldsymbol{f}$ **changes** from **upward** to **downward** or from **downward** to **upward**

Example 1: Look at the graph below and find the open intervals where the graph is concave upward and concave downward.



Solution: Concave up:

Concave down:

Theorem 4.7 (Test for concavity)

Let f be a function whose second derivative exists on an interval I.

1. If $f''(x) > 0$ for all x *in* I, then the graph of f is **concave upward** on **I.**
2. If $f''(x) < 0$ for all x *in* I, then the graph of f is **concave downward** on **I.**

Example 2: (Page 231 Ex 1) Determine the open intervals on which the graph of $f(x) = e^{-x^2/2}$ is concave upward or concave downward.

Solution: $f''(x) = e^{-\frac{x^2}{2}}(x^2 - 1)$. So, $f''(x) = 0$ implies $x = \pm 1$. Since f' is defined on the entire real line we test for the signs f' of on the intervals $(-\infty, -1), (-1, 1)$ *and* $(1, \infty)$

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f'(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	**Concave UP**	**Concave Down**	**Concave UP**

Example 3: (Page 233 Ex 3) Find the point of **inflection** and discuss the **concavity** of $f(x) = x^4 - 4x^3$

Solution: $f''(x) = 12x^2 - 24x$; **setting** $f''(x) = 0$ gives $x = 0$ and $x = 2$. **So,**

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	**Concave UP**	**Concave Down**	**Concave UP**

$f''(x)$ Changes sign at $x = 0$ and $x = 2$, thus $(0, f(0)) = (0, 0)$ and $(2, f(2)) = (2, -16)$ are inflection points

Example 4: Find intervals where the graph is concave up or concave down and all inflection points.

a) $f(x) = x^2 + 10x - 9$

b) $f(x) = x^3 + 3x^2 - 9x + 4$

c) $f(x) = \dfrac{x}{x^2 + 1}$

d) $f(x) = 2x - 5e^{-x}$

The Second Derivative Test

Theorem 4.9 (Second derivative Test)

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum
3. If $f''(c) = 0$, then the Test Fails. That is f can have anything at c

Proof: See text

Example 5: Using the **Second Derivative Test** find the relative extrema for $f(x) = -3x^5 + 5x^3$

Solution: $f'(x) = -15x^4 + 15x^2$. So, $f'(x) = 0$ gives $x = -1, 0, 1$.

$$f''(x) = -60x^3 + 30x$$

Point	$(-1, -2)$	$(1, 2)$	$(0, 0)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(0) = 0$
Conclusion	**Relative Min**	**Relative Max**	**Test Fails**

Example 6: Using the **Second Derivative Test** find the relative extrema

a) $f(x) = x^2 + 10x - 9$

b) $f(x) = x^3 + 3x^2 - 9x + 4$

c) $f(x) = 2x - 5e^{-x}$

Example 7: Use derivatives to determine where the graph is decreasing and concave-up
$f(x) = x^4 - 2x^3$

Example 8: Find where the graph is increasing and concave up. $f(x) = x^3 + 3x^2 - 9x + 4$

Example 9: Find where the graph is increasing and concave up for $f(x) = e^{-x^2}$

Example 10: Determine where the graph is increasing and concave down. $f(x) = \dfrac{x}{x^2 + 1}$

Practice Problems:

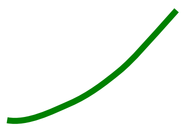
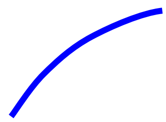
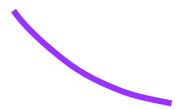
Page 235, Exercises 4.4: 1 – 4, 5, 7, 9, 13, 17, 19, 21, 22, 25, 29, 31, 34, 37, 39, 43, 45, 47, 51, 55, 58, 61, 65, 69

4.6 Curve Sketching (Page 249)

Objectives: By the end of this section students should be able to:

- Analyze and sketch the graph of a function
- Appreciate the role that calculus plays in curve sketching

Summary of derivatives

f'	**(+) increasing**	**(+) increasing**	**(-) decreasing**	**(-) decreasing**
f''	**(+) concave up**	**(-) concave down**	**(+) concave up**	**(-) concave down**
Shape				

Graphs are among the most important tools in understanding concepts in mathematics, in this section we learn how to sketch graphs using information we have studied so far: from college algebra through calculus

Important Guidelines for Curve Sketching

1. Find **domain** and range
2. Find asymptotes if there are any
3. Find symmetry if there are any
4. Find intercepts
5. Find 1st derivative (critical numbers, relative extrema, and intervals of increasing and decreasing)
6. Find 2nd derivative (inflection points, and intervals where graphs are concave up and concave down)
7. Plot intercepts, critical numbers, inflection points, asymptotes, and other points as needed.

Example 1: Sketch the curve $f(x) = x^4 - 2x^3$

Example 2: Sketch the curve $\boldsymbol{f(x) = \frac{2(x^2 - 9)}{x^2 - 4}}$

Practice Problems:

Page 255, Exercises 4.6: 1 – 4, 5, 7, 15, 27, 28, 29, 41, 47, 48, 73, 74, 79 – 82, 89, 90

Optimization Problems (Page 259)

Application of Extrema

Examples:

1. The velocity of a particle (in ft/s) is given by $s(t) = t^2 - 5t + 9$, where ***t*** is the time (in seconds) for which it has traveled. Find the time at which the velocity is at a minimum.
2. Find two numbers whose sum is 110 and whose product is as large as possible. (maximize)
3. Find two non-negative numbers x and y such that their sum is 60 and x^2y is maximized.
4. Find the dimensions of the rectangular field of maximum area that can be made from 240m of fencing material.
5. If the price charged for a bolt is ***p*** cents, then ***x*** thousand bolts will be sold in a certain hardware store, where $p(x) = 9 - \frac{x}{22}$. How many bolts must be sold to maximize revenue?
6. A local club is arranging a charter flight to Hawaii. The cost of the trip is $540 each for 75 passengers, with a refund of $4 per passenger for each passenger in excess of 75. Find the maximum revenue.
7. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is $800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each $10 increase in rent. What is the maximum of revenue?
8. An open box will be made by cutting out a square from each corner of a 3ft by 8ft piece of cardboard and then folding up the sides. The square should be cut from each corner in order to produce a box of maximum volume. What is the maximum volume?
9. A company wishes to manufacture a box with a volume of 40 cubic feet that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material.

Practice Problems:
Page 265, Exercises 4.7: 3, 5, 7, 9, 10, 13, 15, 17 - 20

4.8 Differentials (Page 271)

Objectives: By the end of this section students should be able to

- Understand the concept of tangent line approximation
- Compare values of the differential with actual values
- Estimate errors
- Find the differential of a function using differentiation formulas

Linear Approximation

Differentials

Let $\boldsymbol{y = f(x)}$. When the tangent line to the graph of $\boldsymbol{f}$ at $(\boldsymbol{c}, \boldsymbol{f(c)})$, $\boldsymbol{y = f(c) + f'(c)(x - c)}$ is used as an approximation of the graph of $\boldsymbol{f}$, the quantity $\boldsymbol{\Delta x = x - c}$ is called the change is $\boldsymbol{x}$. When $\boldsymbol{\Delta x}$ is small the change in $\boldsymbol{y}$ is given by $\boldsymbol{\Delta y = f(c + \Delta x) - f(c)}$ can be approximated by $\boldsymbol{\Delta y \approx f'(c)\Delta x}$.

Definition (Differentials)

Let $\boldsymbol{y = f(x)}$ represents a function that is differentiable on an open interval containing $\boldsymbol{x}$. The differential of x (denoted by $\boldsymbol{dx}$) is any non-zero real number. The differential of y (denoted by $\boldsymbol{dy}$) is $\boldsymbol{dy = f'(x)dx}$.

Note: In many type of applications, the differential of y, $\boldsymbol{dy}$, can be used as an approximation of the change in y, $\Delta\boldsymbol{y}$.

Example 1: Find $\boldsymbol{dy}$ *and* $\Delta\boldsymbol{y}$ for the given values of $\boldsymbol{x}$ and $\Delta\boldsymbol{x}$

a) $y = 2x^2 - 5x$; $x = -2,\ \Delta x = 0.2$

b) $y = \dfrac{2x-5}{x+1}$; $x = -2,\ \Delta x = -0.03$

Example 3: Estimation of error

The measured radius of a ball bearing is 0.7 inch. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

Solution: Volume of a sphere is given by $\boldsymbol{V = \frac{4}{3}\pi r^3}$

Given: **r = o.7** in and error $|\ \Delta \boldsymbol{r}\ | = \boldsymbol{0.01}$.

$\boldsymbol{\Delta V \approx dV = 4\pi r^2 dr = 4\pi(0.7)^2(\pm 0.01) \approx \pm 0.06158}$ cubic inch

So, the volume has a propagated error $|\ \boldsymbol{\Delta V}\ |$ $\boldsymbol{of\ about\ 0.06}$ cubic inch.

Relative Error: The Ratio $\frac{dV}{V} = \frac{\boldsymbol{4\pi r^2 dr}}{\boldsymbol{\frac{4}{3}\pi r^3}} = \frac{3\,dr}{r} \approx \frac{3}{0.7}(\pm 0.01) \approx \pm 0.0429$ is called the Relative Error.

The corresponding percent is approximately 4.29%.

Differential Formulas

Let u and v be differentiable function of x.

1. $d(cu) = cdu$ Constant Multiple Rule
2. $d(u \pm v) = du \pm dv$ Sum Difference Rule
3. $d(u\,v) = vdu + udv$ Product Rule
4. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$ Quotient Rule

Proof: 3. $d(u\,v) = (uv)'dx = (u'v + uv')dx = vu'dx + uv'dx = vdu + udv$

Example 3: Find the differential dy of the given functions

a) $y = 5x^3 + 3x$

b) $y = x\cos x$

c) $y = \ln\sqrt{4 - x^2}$

d) $y = e^{2x}\cos 4x$

Example 4: Applications

a) The demand for grass seed (in thousands of pounds) at a price of x dollars is $(x) = -5x^3 - 2x^2 + 1500$. Use the differential to approximate the changes in demand when x changes from \$2 to \$2.50.

b) A company estimates that the revenue (in dollars) from the sale of x doghouses is given by $R(x) = 625 + .03x + .0001x^2$. Use the differential to approximate the change in revenue from the sale of one more doghouse when 1000 doghouses are sold.

Linear Approximation:

A **linear approximation** is an approximation of a general function using a linear function $(\boldsymbol{f(x) = mx + b})$

Let $\boldsymbol{f}$ be a function whose derivative exists. For small nonzero values of Δx, $dy \approx \Delta y$, and

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)dx.$$

This is the linear approximation formula, as you can observe the left hand side is a linear equation of the form $\boldsymbol{mx + b}$.

Example 5: Use differentials to approximate each quantity.

a) $\sqrt{15}$ b) $\sqrt{10}$ c) $\sqrt{115}$ d) $\ln 1.05$

Practice Problems:

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